Nonlinear problems for the square of the fractional Laplacian

ABSTRACT

We study nonlinear problems involving a nonlocal positive operator: the square root of the Laplacian $-\Delta$ in a bounded domain Ω of $\mathbb{R}^n$ with zero Dirichlet boundary condition. We denote this nonlocal operator by $A_{1/2}$, and study nonlinear problems $A_{1/2}u = f(u)$ in Ω and $u = 0$ on $\partial\Omega$ by using variational methods. An important tool in our analysis is to realize our nonlocal problem as a nonlinear local problem in a half cylinder $\mathcal{C} = \Omega \times (0, \infty)$ with a nonlinear Neumann boundary condition on the part $\Omega \times \{0\}$ of the boundary and zero Dirichlet condition on the boundary part $\partial\Omega \times [0, \infty)$. We show a Pohozaev type formula and then we obtain a non-existence result for the critical and supercritical cases $f(u) = u^p$, for $p \geq \frac{n+1}{n-1}$ when Ω is star-shaped. We establish the existence of positive solutions for the subcritical case $1 < p < \frac{n+1}{n-1}$ in a bounded domain and for the critical case with a small perturbation $f(u) = u^{\frac{n+1}{n-1}} + \mu u$, ($\mu > 0$), where we follow the procedure of Brézis-Nirenberg. We show the regularity and an L^∞ estimate of weak solutions. We also obtain a symmetry result of Gidas-Ni-Nirenberg type. Finally by using an analogue of the moving planes method we establish a priori estimates of Gidas-Spruck type for problems with subcritical nonlinearities $f(u) = u^p$ and $1 < p < \frac{n+1}{n-1}$. For this, important ingredients are Liouville theorems for $A_{1/2}$ in the whole space and in the half space. The half space case was unknown, and we prove it by use of the Kelvin transform, the moving planes method and a Hamiltonian identity.