ON DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH DELAY IN ABSTRACT SPACES

VERÓNICA POBLETE

The systematic investigation of delay equations started at the beginning of the twentieth century with the work of Volterra [26]. However, the asymptotic behavior of the solutions was not well understood at that time. In [6], Bátkai and Piazzera, use as the main tools the theory of strongly continuous operator semigroups to develop a general and systematic theory of delay equations with emphasis on the qualitative behavior and their asymptotic properties.

There is exhaustive literature concerning second-order abstract Cauchy problems with delay. Lions [23] and Yosida [30] were the first to use the idea of reduction to first order problems. For more recent literature on well-posedness, see the papers of Engel and Nagel [11], Fattorini [12], Goldstein [15], Krein and Langer [21], Neubrander [25], Xiao and Liang [29]. In recent years, Fourier analysis has become an important tool in the analysis of integro-differential equations with delay. Ever since completion of Fourier's ground-breaking work on the propagation of heat in solid bodies in 1807, followed by the monograph "Théorie Analytique de la Chaleur" in 1822, Fourier analysis has become an indispensable tool in analysis. It is not only essential in the analysis of differential equations, but is also a very important tool in most areas of pure and applied mathematics. It was discovered recently that in analyzing operator equations in abstract spaces, the theory of Fourier multipliers can be used effectively. For operator-valued Fourier multipliers and maximal regularity for evolution equations, contributors include among others Amann [1], Arendt, Batty and Bu [2], Arendt and Bu [3, 4, 5], Clément [9], Denk, Hieber and Prss [10], Girardi, Weis [13, 14], Hytonen [16], Kalton, Lancien [17], Lizama [24], Keyantuo and Lizama [18, 19, 20], Kunstmann and Weis [22], Weis [27, 28]. References to physical problems, most notably visco-elasticity of materials with memory, are found in these works and their respective bibliographies. As far as the structural theory of the Banach spaces (relevant to the theory) and related tools, we mention Bourgain [7] and Burkholder [8]. We turn our attention to second-order problems with delay.

We study equations of the following type

$$u''(t) = Bu'(t) - Au(t) + Gu'_t + Fu_t + f(t)\,, \qquad t \in \mathbb{R} \tag{0.1}$$

where $(A, D(A))$ be a closed linear operator defined on a Banach space X, $u_t(\cdot) = u(t+\cdot)$ and $u'_t(\cdot) = u'(t+\cdot)$ are defined on $[-r, 0]$, $r > 0$. Here $(B, D(B))$ is relatively bounded with respect to the unperturbed operator A with $D(A) \subset D(B)$ and the delay operators F, G are supposed to belong to $\mathcal{B}(C^1([-r, 0], X), X)$.

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In this work we are able to obtain necessary and sufficient conditions in order to guarantee well-posedness of the delay equation (0.1) in the Hölder spaces $C^\alpha(\mathbb{R}, X) \quad (0 < \alpha < 1)$, and under the condition that X is a B-convex space. However we stress that here A is not necessarily the generator of a C_0-semigroup.

We denote by $\mathcal{F}f$ the Fourier transform, i.e.

$$(\mathcal{F}f)(s) := \int_{\mathbb{R}} e^{-ist} f(t)dt$$

$(s \in \mathbb{R}, f \in L^1(\mathbb{R}; X))$.

Definition 0.1. *Let $M : \mathbb{R}\backslash\{0\} \to \mathcal{B}(X,Y)$ be continuous. We say that M is a $\dot{C}^\alpha$ -multiplier if there exists a mapping $L : \dot{C}^\alpha(\mathbb{R}, X) \to \dot{C}^\alpha(\mathbb{R}, Y)$ such that*

$$\int_{\mathbb{R}} (Lf)(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} (\mathcal{F}(\phi \cdot M))(s)f(s)ds \tag{0.2}$$

for all $f \in C^\alpha(\mathbb{R}, X)$ and all $\phi \in C_c^\infty(\mathbb{R}\backslash\{0\})$.

The following multiplier theorem is due to Arendt-Batty and Bu, [2, Theorem 5.3], and play an important role in the proof of our main theorem.

Theorem 0.2. *Let X is B-convex and $M \in C^1(\mathbb{R}\backslash\{0\}, \mathcal{B}(X,Y))$ be such that*

$$\sup_{t\neq 0} ||M(t)|| + \sup_{t\neq 0} ||tM'(t)|| < \infty. \tag{0.3}$$

Then M is a $\dot{C}^\alpha$-multiplier.

Definition 0.3. *We say that (0.1) is C^α-well posed if for each $f \in C^\alpha(\mathbb{R}, X)$ there is a unique function $u \in C^{\alpha+2}(\mathbb{R}, X) \cap C^\alpha(\mathbb{R}, [D(A)])$ such that (0.1) is satisfied.*

Denote by $e_\lambda(t) := e^{i\lambda t}$ for all $\lambda \in \mathbb{R}$, and define the operators $\{F_\lambda\}_{\lambda\in\mathbb{R}}, \{G_\lambda\}_{\lambda\in\mathbb{R}} \subseteq \mathcal{B}(X)$ by

$$F_\lambda x = F(e_\lambda x)\,,\ G_\lambda x = G(e_\lambda x) \text{ for all } \lambda \in \mathbb{R} \text{ and } x \in X. \tag{0.4}$$

We define the *real spectrum* of (0.1) by

$$\sigma(\Delta) = \{s \in \mathbb{R} : s^2 I + isB + isG_s + F_s - A \in \mathcal{B}([D(A)], X) \text{ is not invertible }\}.$$

Our main result in this work is the following theorem.

Theorem 0.4. *Let A be a closed linear operator defined on a B-convex space X. Let B relatively bounded with respect to the unperturbed operator A. Then the following assertions are equivalent*

(i) Equation (0.1) is C^α-well posed.

(ii) $\sigma(\Delta) = \emptyset$ and $\sup_{\eta\in\mathbb{R}} ||i\eta^2(\eta^2 I + i\eta B + i\eta G_\eta + F_\eta - A)^{-1}|| < \infty$.

We denote by $\mathcal{K}_F(X)$ the class of operators in X satisfying (ii) in the above theorem. If $A \in \mathcal{K}_F(X)$ we have $u'', u', Au, Bu, Fu., Gu'_. \in C^\alpha(\mathbb{R}, X)$, and hence we deduce the following result.

Corollary 0.5. *Let X be B-convex and $A \in \mathcal{K}_F(X)$. Then*

(i) (0.1) has a unique solution in $C^{\alpha+2}(\mathbb{R}, X) \cap C^{\alpha}(\mathbb{R}, [D(A)])$ if and only if $f \in C^{\alpha}(\mathbb{R}, X)$.

(ii) There exists a constant $M > 0$ independent of $f \in C^{\alpha}(\mathbb{R}, X)$ such that

$$\|u''\|_{C^\alpha} + \|u'\|_{C^\alpha} + \|Au\|_{C^\alpha} + \|Bu\|_{C^\alpha} + \|Gu'_.\|_{C^\alpha} + \|Fu_.\|_{C^\alpha} \leq M\|f\|_{C^\alpha}$$

where $C^\alpha := C^\alpha(\mathbb{R}, X)$.

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