STRONG CONVERGENCE OF THE CQ METHOD FOR FIXED POINT ITERATION PROCESSES

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Three iteration processes are often used to approximate a fixed point of a nonexpansive mapping T. The first one is introduced by Halpern [7] and is defined as follows: Take an initial guess x_0 arbitrarily and define $\{x_n\}$ recursively by

$$(1) \qquad x_{n+1} = t_n x_0 + (1 - t_n)Tx_n, \quad n \geq 0,$$

where $\{t_n\}_{n=0}^{\infty}$ is a sequence in the interval $[0, 1]$.

The second iteration process is now known as Mann's iteration process [13] which is defined as

$$(2) \qquad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0,$$

where the initial guess x_0 is also taken arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $[0, 1]$.

The third iteration process is referred to as Ishikawa's [8] iteration process which is defined recursively by

$$(3) \qquad \begin{cases} z_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tz_n, \end{cases} \quad n \geq 0,$$

where the initial guess x_0 is taken arbitrarily and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}$ are sequences in the interval $[0, 1]$.

In general not much has been known regarding the convergence of the iteration processes (1)-(3) unless the underlying space X has nice properties which we briefly mention here.

The iteration process (1) has been proved to be strongly convergent in both Hilbert spaces [7, 12, 24] and uniformly smooth Banach spaces [17, 20, 25] whenever the sequence $\{t_n\}$ satisfies the conditions:

(i) $t_n \to 0$;
(ii) $\sum_{n=0}^{\infty} t_n = \infty$; and
(iii) either $\sum_{n=0}^{\infty} |t_n - t_{n+1}| < \infty$ or $\lim_{n\to\infty}(t_n/t_{n+1}) = 1$.

Due to the restriction of condition (ii), the process (1) is widely believed to have slow convergence though the rate of convergence has not been determined. Moreover, Halpern [7] proves that the conditions (i) and (ii) are indeed necessary in the sense that if the process (1) is strongly convergent for all closed convex subsets C of a Hilbert space H and all nonexpansive mappings T on C, then the sequence $\{t_n\}$ must satisfy the conditions (i) and (ii). Thus to improve the rate of convergence of the process (1), one

cannot rely only on the process itself; instead, some additional step of iteration should be taken. One of the purposes of this work is to show that if some appropriate additional step of iteration is performed, then one has strong convergence under the condition (i) only. This would enhance the rate of convergence.

The process (3) is indeed more general than the process (2). But research has been concentrated on the latter due probably to the reasons that the formulation of the process (2) is simpler than that of (3) and that a convergence theorem for the process (2) may possibly lead to a convergence theorem for the process (3) provided the sequence $\{\beta_n\}$ satisfies certain appropriate conditions. However, the introduction of the process (3) has value on its own right. As a matter of fact, the process (2) may fail to converge while the process (3) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space [3]. Both the processes (2) and (3) have only weak convergence, in general (see [4] for an example). For example, Reich [16] shows that if the underlying space X is uniformly convex and has a Frechet differentiable norm and if the sequence $\{\alpha_n\}$ is such that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by the processes (2) converges weakly to a fixed point of T.

An advantage that the process (2) has over the process (1) though the former has only weak convergence in general is the use of the averaged mapping $\alpha I + (1 - \alpha)T$ in each iteration step. This averaged mapping behaves more regularly than the nonexpansive mapping T itself (see [2] for applications of the averaged mappings in inverse problems). The weakness of the process (2) is however its weak convergence.

In order to overcome this weakness, attempts have recently been made to modify the process (2) so that strong convergence is guaranteed. Nakajo and Takahashi [15] proposed the following modification for the process (2) in a Hilbert space H:

$$
(4) \quad
\begin{cases}
x_0 \in C \text{ chosen arbitrarily}, \\
y_n = \alpha_n x_n + (1 - \alpha_n)T x_n, \\
C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\
Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0,
\end{cases}
$$

where P_K denotes the metric projection from H onto a closed convex subset K of H.

Note that we will call the process (4) a CQ method for the Mann iteration process because at each step the Mann iterate (denoted y_n in (4)) is used to construct the sets C_n and Q_n which are in turn used to construct the next iterate x_{n+1} and hence the name.

Nakajo and Takahashi [15] proved that if the sequence $\{\alpha_n\}$ is bounded above away from one, then the sequence $\{x_n\}$ generated by the CQ method (4) converges strongly to $P_{Fix(T)}(x_0)$. An extension of the process (4) to

asymptotically nonexpansive mappings can be found in [11]. See also [10] for another modification of the Mann iteration process (2) which also has strong convergence.

The purpose of this work is to extend Nakajo and Takahashi's iteration process (4) to the Ishikawa iteration process and develop the CQ method to show the strong convergence under the condition (i) only. Finally we develop the CQ method for the proximal point algorithm (PPA) of Rockafellar [18] for finding a zero of a maximal monotone operator in a Hilbert space. Our CQ method has strong convergence under a weaker condition on the error sequence $\{e_n\}$.

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